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**Representation theory for finite groups  
in computer system "CAYLEY"**

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Recently, computational methods are useful for the representation theory, and have been executed by the CAYLEY system by Cannon[1]. In this paper, we will show a usage and some applications of the CAYLEY in the representation theory.

**1. Representation in CAYLEY**

Let  $G$  be a finite group with a set of generators  $\{g_1, \dots, g_l\}$  and  $F$  a splitting field for  $G$  such that the characteristic of  $F$  divides the group order  $|G|$ .

In this paper we treat the action of an element  $g$  of  $G$  on the  $F$ -vector space  $V$  as the product of a vector by a matrix  $V(g)$  on the right. So we can consider the vector space  $V$  as a right  $FG$ -module for the group algebra  $FG$ . In the CAYLEY system, we treat a set  $\{M(g_1), \dots, M(g_l)\}$  as a representation of  $FG$ -module  $V$ . A series of submodules of  $V$

$$0 = V_0 < V_1 < \dots < V_n = V \quad \text{where } V_i/V_{i-1} \text{ is simple}$$

is called a composition series for an  $FG$ -module  $V$ .

**2. The socle**

Let  $\text{Soc}(V)$  denote the socle of  $V$ , namely the sum of all simple  $FG$ -submodules of  $V$ .

LEMMA 1. *Let  $V$  be an  $FG$ -module and  $U$  an  $FG$ -submodule of  $V$  such that  $V/U$  is isomorphic to a simple  $FG$ -module  $W$ . Then the following statements are equivalent.*

- (i) *There is an  $FG$ -submodule  $T$  which is isomorphic to  $W$  and  $\text{Soc}(V) = \text{Soc}(U) \oplus T$ .*
- (ii)  *$V$  is isomorphic to  $U \oplus W$ .*

PROOF: (i)  $\Rightarrow$  (ii). Since  $U \cap T = \text{Soc}(U) \cap T = 0$ ,  $U \oplus T$  is an  $FG$ -submodule of  $V$ . But the dimension of  $V$  is equal to this submodule. So  $V = U \oplus T$ .

(ii)  $\Rightarrow$  (i). Immediate from the definition of the socle.

There is the standard function *composition factor* which is written by Schneider[3] in the CAYLEY system. From Lemma 1, we can get the socle of the  $FG$ -module  $V$  by the following algorithm.

ALGORITHM SOC:

- (1) Let get a composition series  $\{V_i\}_{(i=1,\dots,n)}$  of  $V$  and *socsq* be empty.
- (2) For each  $i$ , see whether  $V_i$  is isomorphic to  $V_{i-1} \oplus V_i/V_{i-1}$  or not. If  $V_i$  can split then append  $V_i/V_{i-1}$  to *socsq*.
- (3) Print *socsq* as the socle of the  $FG$ -module  $V$ .

The main part of this algorithm is investigating that  $V_i$  can split or not. Let  $V$  be an  $FG$ -module and  $U$  an  $FG$ -submodule of  $V$  such that  $V/U$  is isomorphic to a simple  $FG$ -module  $W$ . The dimension of the module  $U$  and the module  $W$  are  $u$  and  $w$ , respectively. In a good basis of  $V$ ,  $V(g)$  is a following matrix

$$\begin{pmatrix} U(g) & 0 \\ D(g) & W(g) \end{pmatrix} \quad \text{for each element } g \text{ of } G$$

where  $D(g)$  is a  $w \times u$ -matrix. Since  $V$  is an  $FG$ -module,  $D$  satisfies a following equation.

$$(*) \quad D(gg') = D(g)U(g') + W(g)D(g') \quad \text{for any } g, g' \text{ in } G$$

The module  $V$  is isomorphic to  $U \oplus W$  if and only if there are some regular matrices  $P$  and

$$(1) \quad PV(g)P^{-1} = \begin{pmatrix} U(g) & 0 \\ 0 & W(g) \end{pmatrix}$$

for all elements  $g$  of  $G$ . What made it difficult is the number of unknowns which have to be processed to find the matrix  $P$ . Thus we prove the next lemma to reduce the number of unknowns.

LEMMA 2. Using the above conditions, the following statements are equivalent.

- (i) There is such a matrix  $P$ .
- (ii) There is a  $w \times u$  matrix  $Q$  such that  $D(g) = W(g)Q - QU(g)$  for any  $g$  in  $G$ .

By Lemma 2, it suffices to find the matrix  $Q$  instead of the matrix  $P$ . So we can reduce the number of unknowns from  $(m+n)^2$  to  $nm$  and see it as the problem of basic linear algebra.

PROOF: (i)  $\Rightarrow$  (ii)

$$\text{Let } P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \quad \text{where } \begin{cases} p_1 : u \times u \text{ matrix} & p_2 : u \times w \text{ matrix} \\ p_3 : w \times u \text{ matrix} & p_4 : w \times w \text{ matrix} \end{cases}$$

Then from (1), we get the following equations for all elements  $g$  of  $G$ .

$$(2) \quad p_1 U(g) + p_2 D(g) = U(g) p_1$$

$$(3) \quad p_2 W(g) = U(g) p_2$$

$$(4) \quad p_3 U(g) + p_4 D(g) = W(g) p_3$$

$$(5) \quad p_4 W(g) = W(g) p_4$$

If matrix  $p_4$  is regular then let  $Q$  be  $p_4^{-1} p_3$ . The matrix  $Q$  satisfies the condition (ii) from (4) and (5).

Since  $W$  is the simple module and we can see that  $p_4$  is an endomorphism of  $FG$ -module  $W$  from (5).

So if the matrix  $p_4$  is not regular then  $p_4$  must be a zero-matrix by Schur's lemma. From the equations (3) and (4),  $p_3 p_2 W(g) = W(g) p_3 p_2$ . If  $p_3 p_2$  is not a zero-matrix then  $p_3 p_2$  is  $\alpha I$  by Schur's lemma where  $\alpha$  is a non-zero element of  $F$  and  $I$  is the unit matrix. The product of (2) and  $\alpha^{-1} p_3$  on the left gives

$$\alpha^{-1} p_3 p_1 U(g) + D(g) = W(g) \alpha^{-1} p_3 p_1$$

by the equation (4). So  $Q$  is  $\alpha^{-1} p_3 p_1$ .

If  $p_3 p_2$  is a zero-matrix then there is a positive integer  $k$  such that  $p_3 p_1^n p_2 = 0$  ( $0 \leq n \leq k$ ) and  $p_3 p_1^{k+1} p_2 \neq 0$  and

$$(2') \quad p_1^{n+1} U(g) = U(g) p_1^{n+1} - \sum_{i=0}^n p_1^i p_2 D(g) p_1^{n-i} \quad \text{for the natural number } n$$

by the easy calculation. When  $n = k$ , the product of (2') and  $p_3$  on the left gives  $p_3 p_1^{k+1} U(g) = W(g) p_3 p_1^{k+1}$  by the equation (4) and  $p_3 p_1^{k+1} p_2 W(g) = W(g) p_3 p_1^{k+1} p_2$  by the equation (3). We can see that  $p_3 p_1^{k+1} p_2$  is  $\alpha I$  by Schur's lemma where  $\alpha$  is a non-zero element of  $F$  and  $I$  is the unit matrix. When  $n = k + 1$ , the product of (2') and  $\alpha^{-1} p_3$  on the left gives

$$\alpha^{-1} p_3 p_1^{k+2} U(g) = W(g) \alpha^{-1} p_3 p_1^{k+2} - D(g)$$

by the equation (4). So  $Q$  is  $\alpha^{-1} p_3 p_1^{k+2}$ .

(ii)  $\Rightarrow$  (i)

Let  $P = \begin{pmatrix} I_m & 0 \\ Q & I_n \end{pmatrix}$  where  $I_m$  and  $I_n$  are the  $m$  and  $n$ -dimensional unit matrix.

Then the matrix  $P$  satisfies the equation (1).

By the way, let think about a  $w \times u$ -matrix  $D(g)$ . Let  $F^{w \times u}$  be a set of  $w \times u$ -matrices over  $F$ ,  $E(W, U)$  a set of map  $D$  from  $G$  to  $F^{w \times u}$  which is satisfies (\*) and  $e(W, U)$  a set of map  $D_Q$  such that  $D_Q(g) = W(g)Q - QU(g)$  where  $Q$  is a  $w \times u$ -matrix. Then  $E(W, U)$  is an  $F$ -space and  $e(W, U)$  an  $F$ -subspace of  $E(W, U)$ . And  $E(W, U)/e(W, U)$  is isomorphic to  $\text{Ext}_{FG}^1(W, U)$  as  $F$ -space. So we can compute the dimension of  $\text{Ext}_{FG}^1(W, U)$  from this equation. In particular,  $E(W, U)$  and  $e(W, U)$  are  $Z^1(G, U)$  and  $B^1(G, U)$  respectively if  $W$  is the trivial module.

### 3. $\Omega^{-1}(M)$

Suppose  $G$  is  $p$ -group. Using  $E(W, U)$ , we can construct the Heller module  $\Omega^{-1}(M)$  of an  $FG$ -module  $M$ . Let  $\bar{E}(M)$  denote  $E(\mathbf{F}, M)/e(\mathbf{F}, M)$  where  $\mathbf{F}$  is the trivial  $FG$ -module and  $\{\bar{d}_i^1\}$  ( $1 \leq i \leq m_1$ ) an  $F$ -basis of  $\bar{E}(M)$ . Then we can make a following representation

$$M_1(g) = \begin{pmatrix} M(g) & & 0 \\ d_1^1(g) & 1 & \\ \vdots & & \ddots \\ d_{m_1}^1(g) & & & 1 \end{pmatrix}$$

where the  $FG$ -module  $M_1$  has  $M$  as a submodule of  $M_1$  and  $M_1/M$  is isomorphic to  $m_1$  copies of the trivial module  $\mathbf{F}$ . Moreover  $\text{Soc}(M) \simeq \text{Soc}(M_1)$ . By the same process, we can make  $FG$ -module  $M_2$  such that  $M_2$  has  $M_1$  as a submodule and  $M_2/M_1$  is isomorphic

to  $m_2(= \dim_F \bar{E}(M_1))$  copies of the trivial module  $\mathbf{F}$  and  $\text{soc}(M_1) \simeq \text{Soc}(M_2)$ . So if we continue this process, then we get

$$M_k(g) = \begin{pmatrix} M(g) & & & & \\ d_1^1(g) & 1 & & 0 & \\ \vdots & & \ddots & & \\ d_{m_1}^1(g) & 0 & & 1 & \\ & d_1^2(g) & & & 1 \\ & \vdots & & \ddots & \\ & d_{m_k}^k(g) & & & 1 \end{pmatrix}$$

as the injective hull of  $M$ . And it's easy to calculate  $\Omega^{-1}(M) = M_k/M$ .

#### 4. Example

Let  $G = \langle x, y, z | x^3 = y^3 = z^3 = (x, z) = (y, z) = 1, (x, y) = z \rangle$  and  $F = GF(3)$  then  $G$  is the extra-special 3-group,  $|G| = 27$  and

$M$	the dimension of $M$	the dimension of the socle series of $M$
$FG$	27	(1, 2, 4, 4, 5, 4, 4, 2, 1)
$\Omega^{-1}(\mathbf{F})$	26	(2, 3, 3, 5, 4, 4, 2, 1)
$\Omega^{-2}(\mathbf{F})$	28	(4, 4, 6, 3, 6, 3, 2)
$\Omega^{-3}(\mathbf{F})$	80	(6, 9, 14, 13, 14, 12, 8, 4)
$\Omega^{-4}(\mathbf{F})$	82	(7, 10, 16, 12, 15, ...)

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